Semiorthogonal decomposition for twisted grassmannians

Sanghoon Baek (KAIST)

Symposium in Algebraic Geometry, Haeundae, Dec., 2012

Overview

- Basic notions: semiorthogonal decomposition, exceptional collection
- Question on exceptional collection, generalization.
- Main results
  - Twisted grassmannians
  - Statement of the theorem
  - Sketch of proof
  - Twisted flags

Semiorthogonal decomposition

$\mathcal{T}$: triangulated category which is linear over a field F.

$\mathcal{S}_i$: full triangulated subcategory of $\mathcal{T}$.

$\mathcal{S}_i^{\perp}$: full subcategory of $\mathcal{T}$ given by $T \in \mathcal{T}$ such that for all $S \in \mathcal{S}_i$ $\mathrm{Hom}_{\mathcal{T}}(S, T) = 0$.

Semiorthogonal decomposition

$\mathcal{T}$: triangulated category which is linear over a field F.

$\mathcal{S}_i$: full triangulated subcategory of $\mathcal{T}$.

$\mathcal{S}_i^{\perp}$: full subcategory of $\mathcal{T}$ given by $T \in \mathcal{T}$ such that for all $S \in \mathcal{S}_i$ $\mathrm{Hom}_{\mathcal{T}}(S, T) = 0$.

A sequence $(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ such that $\langle \mathcal{S}_i, \mathcal{S}_i^{\perp} \rangle = \mathcal{T}$ for all $1 \leq i \leq n$ is called *semiorthogonal* if

$$\mathcal{S}_i \subset \mathcal{S}_j^{\perp} \text{ for all } 1 \leq i < j \leq n.$$

Semiorthogonal decomposition

$\mathcal{T}$: triangulated category which is linear over a field F.

$\mathcal{S}_i$: full triangulated subcategory of $\mathcal{T}$.

$\mathcal{S}_i^{\perp}$: full subcategory of $\mathcal{T}$ given by $T \in \mathcal{T}$ such that for all $S \in \mathcal{S}_i$ $\mathrm{Hom}_{\mathcal{T}}(S, T) = 0$.

A sequence $(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ such that $\langle \mathcal{S}_i, \mathcal{S}_i^{\perp} \rangle = \mathcal{T}$ for all $1 \leq i \leq n$ is called *semiorthogonal* if

$$\mathcal{S}_i \subset \mathcal{S}_j^{\perp} \text{ for all } 1 \leq i < j \leq n.$$

A semiorthogonal sequence $(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ is called a *semiorthogonal decomposition* for $\mathcal{T}$ if $\mathcal{T} = \langle \mathcal{S}_1, \ldots, \mathcal{S}_n \rangle$.

Exceptional collection

A sequence $(E_1, \dots, E_n)$ of obejcts in $\mathcal{T}$ such that for all $1 \le i \le n$

$$\operatorname{Hom}(E_i, E_i[k]) = \begin{cases} 0 & \text{if } k \neq 0 \\ F & \text{otherwise} \end{cases}$$

is called *exceptional* if

$$\operatorname{Hom}(E_j, E_i[k]) = 0 \text{ for all } 1 \le i < j \le n \text{ and all } k.$$

Exceptional collection

A sequence $(E_1, \ldots, E_n)$ of obejcts in $\mathcal{T}$ such that for all $1 \leq i \leq n$

$$\mathrm{Hom}(E_i, E_i[k]) = \begin{cases} 0 & \text{if } k \neq 0 \\ F & \text{otherwise} \end{cases}$$

is called <u>*exceptional*</u> if

$$\mathrm{Hom}(E_j, E_i[k]) = 0 \text{ for all } 1 \leq i < j \leq n \text{ and all } k.$$

An exceptional sequence $(E_1, \ldots, E_n)$ is said to be <u>*full*</u> if $\mathcal{T} = \langle E_1, \ldots, E_n \rangle$.

Examples: semiorthogonal decomposition

(i) Any full traingulated subcategory $\mathcal{S} \subset \mathcal{T}$ defines a semiorthogonal decomposition for $\mathcal{T}$ if $\langle \mathcal{S}, \mathcal{S}^{\perp} \rangle = \mathcal{T}$.

Examples: semiorthogonal decomposition

(i) Any full traingulated subcategory $\mathcal{S} \subset \mathcal{T}$ defines a semiorthogonal decomposition for $\mathcal{T}$ if $\langle \mathcal{S}, \mathcal{S}^{\perp} \rangle = \mathcal{T}$.

(ii) Let $(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ be a sequence of full traingulated subcategories of $\mathcal{T}$ such that $\mathcal{S}_i \subset \mathcal{S}_j^{\perp}$ for all $1 \leq i < j \leq n$. If the sequence generates $\mathcal{T}$, then this sequence defines a semiorthogonal decomposition for $\mathcal{T}$ (without assuming the condition $\langle \mathcal{S}_i, \mathcal{S}_i^{\perp} \rangle = \mathcal{T}$).

Examples: semiorthogonal decomposition

(i) Any full traingulated subcategory $\mathcal{S} \subset \mathcal{T}$ defines a semiorthogonal decomposition for $\mathcal{T}$ if $\langle \mathcal{S}, \mathcal{S}^{\perp} \rangle = \mathcal{T}$.

(ii) Let $(\mathcal{S}_1, \dots, \mathcal{S}_n)$ be a sequence of full traingulated subcategories of $\mathcal{T}$ such that $\mathcal{S}_i \subset \mathcal{S}_j^{\perp}$ for all $1 \leq i < j \leq n$. If the sequence generates $\mathcal{T}$, then this sequence defines a semiorthogonal decomposition for $\mathcal{T}$ (without assuming the condition $\langle \mathcal{S}_i, \mathcal{S}_i^{\perp} \rangle = \mathcal{T}$).

(iii) Let $(E_1, \dots, E_n)$ be a (full) exceptional collection in $\mathcal{T}$. Then, the seq.

$$(\langle E_1 \rangle, \dots, \langle E_n \rangle)$$

gives a semiorthogonal seq. (decomposition).

Examples: exceptional collection

$\mathcal{T} = \mathrm{D}(X) :=$ the bounded derived category of coherent sheaves on a scheme X. For any $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \mathcal{T}$,

$$\mathrm{Hom}_{\mathcal{T}}(\mathcal{F}^\bullet, \mathcal{G}^\bullet[k]) = \mathrm{Ext}^k(\mathcal{F}^\bullet, \mathcal{G}^\bullet).$$

Examples: exceptional collection

$\mathcal{T} = \mathrm{D}(X) :=$ the bounded derived category of coherent sheaves on a scheme X. For any $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \mathcal{T}$,

$$\mathrm{Hom}_{\mathcal{T}}(\mathcal{F}^\bullet, \mathcal{G}^\bullet[k]) = \mathrm{Ext}^k(\mathcal{F}^\bullet, \mathcal{G}^\bullet).$$

(i) (Beilinson) Let $X = \mathbb{P}^n_F = \mathbb{P}(V)$. Then,

$$(\mathscr{O}(-n), \mathscr{O}(-n+1), \ldots, \mathscr{O}(-1), \mathscr{O})$$

is a full exceptional sequence:

Examples: exceptional collection

$\mathcal{T} = \mathrm{D}(X) :=$ the bounded derived category of coherent sheaves on a scheme X. For any $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \mathcal{T}$,

$$\mathrm{Hom}_{\mathcal{T}}(\mathcal{F}^\bullet, \mathcal{G}^\bullet[k]) = \mathrm{Ext}^k(\mathcal{F}^\bullet, \mathcal{G}^\bullet).$$

(i) (Beilinson) Let $X = \mathbb{P}^n_F = \mathbb{P}(V)$. Then,

$$(\mathscr{O}(-n), \mathscr{O}(-n+1), \dots, \mathscr{O}(-1), \mathscr{O})$$

is a full exceptional sequence:

- $\mathrm{Ext}^k(\mathscr{O}(j), \mathscr{O}(i)) = H^k(X, \mathscr{O}(i-j)) = F$ (if $i = j$, $k = 0$), or 0.

Examples: exceptional collection

$\mathcal{T} = \mathrm{D}(X) :=$ the bounded derived category of coherent sheaves on a scheme X. For any $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \mathcal{T}$,

$$\mathrm{Hom}_{\mathcal{T}}(\mathcal{F}^\bullet, \mathcal{G}^\bullet[k]) = \mathrm{Ext}^k(\mathcal{F}^\bullet, \mathcal{G}^\bullet).$$

(i) (Beilinson) Let $X = \mathbb{P}^n_F = \mathbb{P}(V)$. Then,

$$(\mathscr{O}(-n), \mathscr{O}(-n+1), \dots, \mathscr{O}(-1), \mathscr{O})$$

is a full exceptional sequence:

- $\mathrm{Ext}^k(\mathscr{O}(j), \mathscr{O}(i)) = H^k(X, \mathscr{O}(i-j)) = F$ (if $i = j$, $k = 0$), or 0.
- Let $\mathscr{E} := \mathscr{O}(1) \boxtimes Q$, where $0 \to \mathscr{O}(-1) \to V \otimes \mathscr{O} \to Q \to 0$.
  For $id_V = s \in H^0(X \times X, \mathscr{E})$, we have $Z(s) = \Delta \subset X \times X$ and

$$0 \to \wedge^n(\mathscr{E}^*) \to \wedge^{n-1}(\mathscr{E}^*) \to \cdots \to \mathscr{E}^* \to \mathscr{O}_{X \times X} \to \mathscr{O}_\Delta \to 0. \quad (1)$$

For $\mathscr{F} \in \mathrm{D}(X \times X)$, define $\Phi(\mathscr{F}) : \mathcal{T} \to \mathcal{T}$ by

$$\mathscr{G} \mapsto (\pi_1)_*(\pi_2^*\mathscr{G} \otimes \mathscr{F}).$$

Then, for any $\mathscr{H} \in \mathcal{T}$, we have

$$\Phi(\mathscr{O}_\Delta)(\mathscr{H}) = \mathscr{H},\ \Phi(\wedge^i(\mathscr{E}^*))(\mathscr{H}) = H^\bullet(X, \mathscr{H} \otimes \Omega^i(i)) \otimes \mathscr{O}(-i).$$

As Φ is exact, the result follows from (1).

For $\mathscr{F} \in \mathrm{D}(X \times X)$, define $\Phi(\mathscr{F}) : \mathcal{T} \to \mathcal{T}$ by

$$\mathscr{G} \mapsto (\pi_1)_*(\pi_2^*\mathscr{G} \otimes \mathscr{F}).$$

Then, for any $\mathscr{H} \in \mathcal{T}$, we have

$$\Phi(\mathscr{O}_\Delta)(\mathscr{H}) = \mathscr{H},\ \Phi(\wedge^i(\mathscr{E}^*))(\mathscr{H}) = H^\bullet(X, \mathscr{H} \otimes \Omega^i(i)) \otimes \mathscr{O}(-i).$$

As Φ is exact, the result follows from (1).

(ii) Exceptional collection need not always exist: for instance, if X is a smooth projective variety of $\dim(X) = n$ with trivial canonical class, then

$$F = \mathrm{Hom}_{\mathcal{T}}(E, E) = \mathrm{Ext}^n(E, E)^* = 0.$$

Question : exceptional collection

Kapranov constructed full exceptional collections on Grassmannians and projective quadrics.

Question

Does any projective homogeneous variety G/P under a split semi simple algebraic group G admit a full exceptional collection?

Question : exceptional collection

Kapranov constructed full exceptional collections on Grassmannians and projective quadrics.

Question

Does any projective homogeneous variety G/P under a split semi simple algebraic group G admit a full exceptional collection?

- Type of $G = A_n, G_2$: full exceptional collections were constructed by Kapranov, Kuznetsov.
- Types of $G = B_n, C_n, D_n$: full exceptional collections for $P = P_1, P_2$ were constructed by Kapranov, Kuznetsov.
- Type of $G = E_6, E_7, E_8, F_4$: this is completely open.

Generalization: semiorthogonal decomposition

- Orlov and Kuznetsov generalized Kapronov's results on grassmannians and quadrics to semiorthogonal decompositions, respectively.

E.g. Given a projective bundle $p : \mathbb{P}(\mathscr{E}) \to X$ associated to a vector bundle $\mathscr{E}$ over X of rank $n+1$, the sequence

$$(\mathrm{D}(X) \otimes \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-n), \ldots, \mathrm{D}(X) \otimes \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-1), \mathrm{D}(X))$$

gives a semiorthogonal decomposition for $\mathbb{P}(\mathscr{E})$.

Generalization: semiorthogonal decomposition

- Orlov and Kuznetsov generalized Kapronov's results on grassmannians and quadrics to semiorthogonal decompositions, respectively.

E.g. Given a projective bundle $p : \mathbb{P}(\mathscr{E}) \to X$ associated to a vector bundle $\mathscr{E}$ over X of rank $n+1$, the sequence

$$(\mathrm{D}(X) \otimes \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-n), \ldots, \mathrm{D}(X) \otimes \mathscr{O}_{\mathbb{P}(\mathscr{E})}(-1), \mathrm{D}(X))$$

gives a semiorthogonal decomposition for $\mathbb{P}(\mathscr{E})$.

- Bernardara extended Orlov's result on projective bundles to the twisted forms.

- The goal of this talk is to extend Orlov's result on grassmannian bundles to the twisted forms.

Twisted grassmannians

X : Noetherian scheme over a field of char. 0

$\mathscr{A}$: sheaf of Azumaya algebras of rank n^2 over X

Twisted grassmannians

X : Noetherian scheme over a field of char. 0

$\mathscr{A}$: sheaf of Azumaya algebras of rank n^2 over X

For $1 \leq k < n$, a <u>*twisted grassmannian*</u> $p : \mathrm{Gr}(k, \mathscr{A}) \to X$ is defined by the representable functor

$$(Y \xrightarrow{\phi} X) \mapsto \left\{ \begin{array}{l} \text{sheaves of left ideals } \mathscr{I} \text{ of } \phi^*\mathscr{A} \mid \phi^*\mathscr{A}/\mathscr{I} \text{ is a} \\ \text{locally free } \mathscr{O}_Y\text{-modules of rank } n(n-k) \end{array} \right\}.$$

Twisted grassmannians

X : Noetherian scheme over a field of char. 0

$\mathscr{A}$: sheaf of Azumaya algebras of rank n^2 over X

For $1 \leq k < n$, a *twisted grassmannian* $p : \mathrm{Gr}(k, \mathscr{A}) \to X$ is defined by the representable functor

$$(Y \xrightarrow{\phi} X) \mapsto \left\{ \begin{array}{l} \text{sheaves of left ideals } \mathscr{I} \text{ of } \phi^*\mathscr{A} \mid \phi^*\mathscr{A}/\mathscr{I} \text{ is a} \\ \text{locally free } \mathscr{O}_Y\text{-modules of rank } n(n-k) \end{array} \right\}.$$

$\exists$ étale covering $i : U \to X$ and a locally free sheaf $\mathscr{E}$ of rank n over U with the following pullback diagram

$$\begin{array}{ccc} \mathrm{Gr}(k, \mathscr{E}) \simeq \mathrm{Gr}(k, \mathscr{E}nd(\mathscr{E})) & \xrightarrow{j} & \mathrm{Gr}(k, \mathscr{A}) \\ \downarrow q & & \downarrow p \\ U & \xrightarrow{i} & X. \end{array}$$

Consider the tautological exact sequence of sheaves on $\mathrm{Gr}(k, \mathscr{E})$

$$0 \to \mathscr{R} \to q^*\mathscr{E} \to \mathscr{Q} \to 0,$$

where $\operatorname{rank}(\mathscr{R}) = k$.

Consider the tautological exact sequence of sheaves on $\mathrm{Gr}(k, \mathscr{E})$

$$0 \to \mathscr{R} \to q^*\mathscr{E} \to \mathscr{Q} \to 0,$$

where $\mathrm{rank}(\mathscr{R}) = k$.

For a partition $\alpha = (\alpha_1, \cdots, \alpha_k)$ with $0 \leq \alpha_i \leq n - k$, we denoted by S^α the Schur functor for α.

E.g. ▪ If V is a k-dimensional vector space, then $S^\alpha V$ is the irreducible representation of $\mathrm{GL}(V)$ with the highest weight α.

▪ For $n = 4$ and $k = 2$, we have $S^{(i,0)}\mathscr{R} = \mathrm{Sym}^i \mathscr{R}$, $S^{(1,1)}\mathscr{R} = \wedge^2\mathscr{R}$, $S^{(2,1)}\mathscr{R} = \mathscr{R} \otimes \wedge^2\mathscr{R}$, and $S^{(2,2)}\mathscr{R} = \wedge^2\mathscr{R} \otimes \wedge^2\mathscr{R}$.

Main result

Define $S(\alpha)$ to be the full subcategory of $\mathrm{D}(\mathrm{Gr}(k, \mathscr{A}))$ generated by $\mathscr{M}$ in $\mathrm{D}(\mathrm{Gr}(k, \mathscr{A}))$ satisfying

$$\mathscr{M}|_{\mathrm{Gr}(k,\mathscr{E})} \simeq q^*\mathscr{N} \otimes S^{\alpha}\mathscr{R},$$

for some $\mathscr{N} \in \mathrm{D}(U)$.

Main result

Define $S(\alpha)$ to be the full subcategory of $\mathrm{D}(\mathrm{Gr}(k, \mathscr{A}))$ generated by $\mathscr{M}$ in $\mathrm{D}(\mathrm{Gr}(k, \mathscr{A}))$ satisfying

$$\mathscr{M}|_{\mathrm{Gr}(k,\mathscr{E})} \simeq q^*\mathscr{N} \otimes S^\alpha \mathscr{R},$$

for some $\mathscr{N} \in \mathrm{D}(U)$.

Theorem

Let $(S(\alpha) \,|\, \alpha = (\alpha_1, \cdots, \alpha_k), 0 \leq \alpha_i \leq n-k)$ be a sequence of the full subcategories of $\mathrm{D}(\mathrm{Gr}(k, \mathscr{A}))$ by the lexicographical order on α. Then this sequence gives a semiorthogonal decomposition of $D(\mathrm{Gr}(k, \mathscr{A}))$.

Proof of theorem

Let $\alpha \neq \alpha'$ with $0 \leq \alpha_i, \alpha'_i \leq n - k$.

Claim 1: $RHom(\mathscr{M}, \mathscr{M}') = 0$ for $\mathscr{M} \in S(\alpha)$ and $\mathscr{M}' \in S(\alpha')$.

Proof of theorem

Let $\alpha \neq \alpha'$ with $0 \leq \alpha_i, \alpha'_i \leq n - k$.

<u>Claim 1</u>: $RHom(\mathscr{M}, \mathscr{M}') = 0$ for $\mathscr{M} \in S(\alpha)$ and $\mathscr{M}' \in S(\alpha')$.

By the local to global Ext spectral sequence, it's enough to show that $R\mathscr{H}om(\mathscr{M}, \mathscr{M}') = 0$.

Let $\mathscr{M}|_{\mathrm{Gr}(k,\mathscr{E})} \simeq q^*\mathscr{N} \otimes S^{\alpha}\mathscr{R}$ and $\mathscr{M}'|_{\mathrm{Gr}(k,\mathscr{E})} \simeq q^*\mathscr{N}' \otimes S^{\alpha'}\mathscr{R}$.

Proof of theorem

Let $\alpha \neq \alpha'$ with $0 \leq \alpha_i, \alpha'_i \leq n - k$.

Claim 1: $RHom(\mathscr{M}, \mathscr{M}') = 0$ for $\mathscr{M} \in S(\alpha)$ and $\mathscr{M}' \in S(\alpha')$.

By the local to global Ext spectral sequence, it's enough to show that $R\mathscr{H}om(\mathscr{M}, \mathscr{M}') = 0$.

Let $\mathscr{M}|_{\mathrm{Gr}(k,\mathscr{E})} \simeq q^*\mathscr{N} \otimes S^{\alpha}\mathscr{R}$ and $\mathscr{M}'|_{\mathrm{Gr}(k,\mathscr{E})} \simeq q^*\mathscr{N}' \otimes S^{\alpha'}\mathscr{R}$.

By the Littlewood-Richardson rule, we have

$$R\mathscr{H}om(S^{\alpha}\mathscr{R}, S^{\alpha'}\mathscr{R}) = S^{\alpha'}\mathscr{R} \otimes (S^{\alpha}\mathscr{R})^* = \bigoplus n_\beta \cdot S^{\beta}\mathscr{R},$$

where β is of the form $(\beta_1, \cdots, \beta_k)$, with $-(n-k) \leq \beta_i \leq n - k$.

By the Borel-Bott-Weil theorem, we have $H^0(\mathrm{Gr}(k, \mathscr{E}), S^{\beta}\mathscr{R}) = 0$.

Proof of theorem

Therefore, we have

$$Rq_*(\mathscr{H}om(S^\alpha\mathscr{R}, S^{\alpha'}\mathscr{R})) = 0. \tag{2}$$

It is enough to show the result locally. By the adjoint property of Rq_* and q^*, projection formula, and (2), we have

$$\begin{aligned}
&R\mathscr{H}om(\mathscr{M}|_{\mathrm{Gr}(k,\mathscr{E})}, \mathscr{M}'|_{\mathrm{Gr}(k,\mathscr{E})}) \\
&= R\mathscr{H}om(q^*\mathscr{N}, q^*\mathscr{N}' \otimes \mathscr{H}om(S^\alpha\mathscr{R}, S^{\alpha'}\mathscr{R})) \\
&= R\mathscr{H}om(\mathscr{N}, Rq_*(q^*\mathscr{N}' \otimes \mathscr{H}om(S^\alpha\mathscr{R}, S^{\alpha'}\mathscr{R}))) \\
&= R\mathscr{H}om(\mathscr{N}, \mathscr{N}' \otimes Rq_*(\mathscr{H}om(S^\alpha\mathscr{R}, S^{\alpha'}\mathscr{R}))) \\
&= 0.
\end{aligned}$$

Proof of theorem

Claim 2: $(S(\alpha))$ generates $\mathrm{D}(\mathrm{Gr}(k, \mathscr{A}))$.

$\exists$ sheaves $\mathscr{F}_\alpha$ of right $\mathscr{A}^{\otimes|\alpha|}$-modules and sheaves $\mathscr{G}_{\alpha^*}$ of left $\mathscr{A}^{\otimes|\alpha|}$-modules such that

$$j^*\mathscr{F}_\alpha \simeq S^\alpha\mathscr{R} \otimes q^*((\mathscr{E}^*)^{\otimes|\alpha|}),\ j^*\mathscr{G}_{\alpha^*} \simeq q^*(\mathscr{E}^{\otimes|\alpha|}) \otimes S^{\alpha^*}\mathscr{Q}^*.$$

Moreover, the sequence

$\mathscr{R} \boxtimes \mathscr{Q}^* \to \mathcal{O}_{\mathrm{Gr}(k,\mathscr{E})\times\mathrm{Gr}(k,\mathscr{E})} \to \mathcal{O}_{\Delta(\mathrm{Gr}(k,\mathscr{E})/U)} \to 0$ descends to the sequence $\mathscr{F}_{(1)} \boxtimes \mathscr{G}_{(1)} \to \mathcal{O}_{\mathrm{Gr}(k,\mathscr{A})\times\mathrm{Gr}(k,\mathscr{A})} \to \mathcal{O}_{\Delta(\mathrm{Gr}(k,\mathscr{A})/X)} \to 0.$

Proof of theorem

Claim 2: $(S(\alpha))$ generates $\mathrm{D}(\mathrm{Gr}(k,\mathscr{A}))$.

$\exists$ sheaves $\mathscr{F}_\alpha$ of right $\mathscr{A}^{\otimes|\alpha|}$-modules and sheaves $\mathscr{G}_{\alpha^*}$ of left $\mathscr{A}^{\otimes|\alpha|}$-modules such that

$$j^*\mathscr{F}_\alpha \simeq S^\alpha\mathscr{R} \otimes q^*((\mathscr{E}^*)^{\otimes|\alpha|}),\ j^*\mathscr{G}_{\alpha^*} \simeq q^*(\mathscr{E}^{\otimes|\alpha|}) \otimes S^{\alpha^*}\mathscr{Q}^*.$$

Moreover, the sequence

$\mathscr{R} \boxtimes \mathscr{Q}^* \to \mathcal{O}_{\mathrm{Gr}(k,\mathscr{E})\times \mathrm{Gr}(k,\mathscr{E})} \to \mathcal{O}_{\Delta(\mathrm{Gr}(k,\mathscr{E})/U)} \to 0$ descends to the sequence $\mathscr{F}_{(1)} \boxtimes \mathscr{G}_{(1)} \to \mathcal{O}_{\mathrm{Gr}(k,\mathscr{A})\times \mathrm{Gr}(k,\mathscr{A})} \to \mathcal{O}_{\Delta(\mathrm{Gr}(k,\mathscr{A})/X)} \to 0$.

Hence, we have the Koszul resolution:

$$0 \to \Lambda^{k(n-k)}(\mathscr{F}_{(1)} \boxtimes \mathscr{G}_{(1)}) \to \Lambda^{k(n-k)-1}(\mathscr{F}_{(1)} \boxtimes \mathscr{G}_{(1)}) \to \cdots$$

$$\cdots \to \mathscr{F}_{(1)} \boxtimes \mathscr{G}_{(1)} \to \mathcal{O}_{\mathrm{Gr}(k,\mathscr{A})\times_X \mathrm{Gr}(k,\mathscr{A})} \to \mathcal{O}_{\Delta(\mathrm{Gr}(k,\mathscr{A})/X)} \to 0.$$

Proof of theorem

As $\Lambda^m(\mathscr{F}_{(1)} \boxtimes \mathscr{G}_{(1)}) = \bigoplus_{|\alpha|=m} \mathscr{F}_\alpha \boxtimes \mathscr{G}_{\alpha^*}$ for $1 \leq m \leq k(n-k)$,

$$\mathcal{O}_{\Delta(\mathrm{Gr}(k,\mathscr{A})/X)} \in \langle \pi_1^* \mathscr{F}_\alpha \otimes \pi_2^* \mathscr{G}_{\alpha^*} \mid 0 \leq |\alpha| \leq k(n-k) \rangle$$

of $D(\mathrm{Gr}(k,\mathscr{A}) \times \mathrm{Gr}(k,\mathscr{A}))$.

Proof of theorem

As $\Lambda^m(\mathscr{F}_{(1)} \boxtimes \mathscr{G}_{(1)}) = \bigoplus_{|\alpha|=m} \mathscr{F}_\alpha \boxtimes \mathscr{G}_{\alpha^*}$ for $1 \leq m \leq k(n-k)$,

$$\mathcal{O}_{\Delta(\mathrm{Gr}(k,\mathscr{A})/X)} \in \langle \pi_1^* \mathscr{F}_\alpha \otimes \pi_2^* \mathscr{G}_{\alpha^*} \mid 0 \leq |\alpha| \leq k(n-k) \rangle$$

of $D(\mathrm{Gr}(k,\mathscr{A}) \times \mathrm{Gr}(k,\mathscr{A}))$.

Since we have $\mathscr{M} = R(\pi_1)_*(\pi_2^*\mathscr{M} \otimes \mathcal{O}_{\Delta(\mathrm{Gr}(k,\mathscr{A})/X)})$ for any $\mathscr{M} \in D(\mathrm{Gr}(k,\mathscr{A}))$, it is enough to verify that

$$R(\pi_1)_*\left(\pi_2^*\mathscr{M} \otimes (\pi_1^*\mathscr{F}_\alpha \otimes \pi_2^*\mathscr{G}_{\alpha^*})\right) \in S(\alpha):$$

$R(\pi_1)_*\left(\pi_2^*\mathscr{M} \otimes (\pi_1^*\mathscr{F}_\alpha \otimes \pi_2^*\mathscr{G}_{\alpha^*})\right) = R(\pi_1)_*\left(\pi_2^*(\mathscr{M} \otimes \mathscr{G}_{\alpha^*})\right) \otimes \mathscr{F}_\alpha$, this is isomorphic to

$$q^*\left(Rq_*(\mathscr{M} \otimes S^{\alpha^*}\mathscr{Q})\right) \otimes S^\alpha \mathscr{R} \text{ over } \mathrm{Gr}(k,\mathscr{E}). \quad \square$$

Twisted flags

Let $1 \leq k_1 < \cdots < k_m < n$ be a sequence of integers.

We denote by $\mathrm{Fl}(k_1, \cdots, k_m, \mathscr{A})$ the functor defined by $(Y \xrightarrow{\phi} X) \mapsto$ the set of sheaves of left ideals $\mathscr{I}_1 \subset \cdots \subset \mathscr{I}_m$ of $\phi^*\mathscr{A}$ such that $\phi^*\mathscr{A}/\mathscr{I}_i$ is a locally free $\mathscr{O}_Y$-modules of rank $n(n-k_i)$.

Twisted flags

Let $1 \leq k_1 < \cdots < k_m < n$ be a sequence of integers.

We denote by $\mathrm{Fl}(k_1, \cdots, k_m, \mathscr{A})$ the functor defined by $(Y \xrightarrow{\phi} X) \mapsto$ the set of sheaves of left ideals $\mathscr{I}_1 \subset \cdots \subset \mathscr{I}_m$ of $\phi^*\mathscr{A}$ such that $\phi^*\mathscr{A}/\mathscr{I}_i$ is a locally free $\mathscr{O}_Y$-modules of rank $n(n-k_i)$.

$\exists$ an étale covering $i : U \to X$ and a locally free sheaf $\mathscr{E}$ of rank n over U with the following pullback diagram

$$\begin{array}{ccc} \mathrm{Fl}(k_1, \cdots, k_m, \mathscr{E}nd(\mathscr{E})) & \xrightarrow{j} & \mathrm{Fl}(k_1, \cdots, k_m, \mathscr{A}) \\ \downarrow q & & \downarrow p \\ U & \xrightarrow{i} & X. \end{array}$$

We have the tautological flags

$$\mathscr{R}_1 \hookrightarrow \cdots \hookrightarrow \mathscr{R}_m \hookrightarrow q^*\mathscr{E} \twoheadrightarrow \mathscr{Q}_1 \twoheadrightarrow \cdots \twoheadrightarrow \mathscr{Q}_m,$$

where $\operatorname{rank}(\mathscr{R}_i) = k_i$ and $\operatorname{rank}(\mathscr{Q}_i) = n - k_i$.

We have the tautological flags

$$\mathscr{R}_1 \hookrightarrow \cdots \hookrightarrow \mathscr{R}_m \hookrightarrow q^*\mathscr{E} \twoheadrightarrow \mathscr{Q}_1 \twoheadrightarrow \cdots \twoheadrightarrow \mathscr{Q}_m,$$

where $\mathrm{rank}(\mathscr{R}_i) = k_i$ and $\mathrm{rank}(\mathscr{Q}_i) = n - k_i$.

Let $\alpha(1), \cdots, \alpha(m-1), \alpha(m)$ be partitions of the forms

$$(\alpha_1, \cdots, \alpha_{k_1}), \cdots, (\alpha_1, \cdots, \alpha_{k_{m-1}}), (\alpha_1, \cdots, \alpha_{k_m})$$

with $0 \le \alpha_i \le k_2 - k_1, \cdots, 0 \le \alpha_i \le k_m - k_{m-1}, 0 \le \alpha_i \le n - k_m$, respectively.

We have the tautological flags

$$\mathscr{R}_1 \hookrightarrow \cdots \hookrightarrow \mathscr{R}_m \hookrightarrow q^*\mathscr{E} \twoheadrightarrow \mathscr{Q}_1 \twoheadrightarrow \cdots \twoheadrightarrow \mathscr{Q}_m,$$

where $\mathrm{rank}(\mathscr{R}_i) = k_i$ and $\mathrm{rank}(\mathscr{Q}_i) = n - k_i$.

Let $\alpha(1), \cdots, \alpha(m-1), \alpha(m)$ be partitions of the forms

$$(\alpha_1, \cdots, \alpha_{k_1}), \cdots, (\alpha_1, \cdots, \alpha_{k_{m-1}}), (\alpha_1, \cdots, \alpha_{k_m})$$

with $0 \leq \alpha_i \leq k_2 - k_1, \cdots, 0 \leq \alpha_i \leq k_m - k_{m-1}, 0 \leq \alpha_i \leq n - k_m$, respectively.

We define $S(\alpha(1), \cdots, \alpha(m))$ to be the full subcategory of $\mathrm{D}(\mathrm{Fl}(k_1, \cdots, k_m, \mathscr{A}))$ generated by $\mathscr{M}$ in $\mathrm{D}(\mathrm{Fl}(k_1, \cdots, k_m, \mathscr{A}))$ satisfying

$$\mathscr{M}|_{\mathrm{Fl}(k_1, \cdots, k_m, \mathscr{E})} \simeq q^*\mathscr{N} \otimes S^{\alpha(1)}\mathscr{R}_1 \otimes \cdots \otimes S^{\alpha(m)}\mathscr{R}_m,$$

for some $\mathscr{N} \in \mathrm{D}(U)$.

Corollary

Let $(S(\alpha(1), \cdots, \alpha(m)) \,|\, \forall$ *partitions of the form* $\alpha(i), 1 \leq i \leq m)$ *be a sequence of the full subcategories of* $\mathrm{D}(\mathrm{FI}(k_1, \cdots, k_m, \mathscr{A}))$ *in lexicographical order. Then this gives a semiorthogonal decomposition of* $\mathrm{D}(\mathrm{FI}(k_1, \cdots, k_m, \mathscr{A}))$*.*

Corollary

Let $(S(\alpha(1), \cdots, \alpha(m)) \mid \forall$ *partitions of the form* $\alpha(i), 1 \leq i \leq m)$ *be a sequence of the full subcategories of* $\mathrm{D}(\mathrm{FI}(k_1, \cdots, k_m, \mathscr{A}))$ *in lexicographical order. Then this gives a semiorthogonal decomposition of* $\mathrm{D}(\mathrm{FI}(k_1, \cdots, k_m, \mathscr{A}))$.

Proof

Induction on m. Assume that the result holds for $m-1$. There are projections

$$\mathrm{FI}(k_1, \cdots, k_m, \mathscr{E}) \xrightarrow{q_m} \cdots \xrightarrow{q_2} \mathrm{FI}(k_m, \mathscr{E}) \xrightarrow{q_1} U$$

and

$$\mathrm{FI}(k_1, \cdots, k_m, \mathscr{A}) \xrightarrow{p_m} \cdots \xrightarrow{p_2} \mathrm{FI}(k_m, \mathscr{A}) \xrightarrow{p_1} X.$$

Let $\mathscr{R}_2' \subset (q_1 \circ \cdots \circ q_{m-1})^*\mathscr{E}$ be the tautological subsheaf over $\mathrm{Fl}(k_2, \cdots, k_m, \mathscr{E})$.

Let $\mathscr{A}'$ be the sheaf of Azumaya algebra over $\mathrm{Fl}(k_2, \cdots, k_m, \mathscr{A})$ from $\mathscr{E}nd(\mathscr{R}_2')$ by descent.

Let $\mathscr{R}_2' \subset (q_1 \circ \cdots \circ q_{m-1})^* \mathscr{E}$ be the tautological subsheaf over $\mathrm{Fl}(k_2, \cdots, k_m, \mathscr{E})$.

Let $\mathscr{A}'$ be the sheaf of Azumaya algebra over $\mathrm{Fl}(k_2, \cdots, k_m, \mathscr{A})$ from $\mathscr{E}nd(\mathscr{R}_2')$ by descent.

Then, we have

$$\mathrm{Fl}(k_1, \cdots, k_m, \mathscr{E}) = \mathrm{Gr}_{\mathrm{Fl}(k_2, \cdots, k_m, \mathscr{E})}(k_1, \mathscr{R}_2')$$

and

$$\mathrm{Fl}(k_1, \cdots, k_m, \mathscr{A}) = \mathrm{Gr}_{\mathrm{Fl}(k_2, \cdots, k_m, \mathscr{A})}(k_1, \mathscr{A}').$$

Now the result follows from the proof Theorem. □

Thank you.